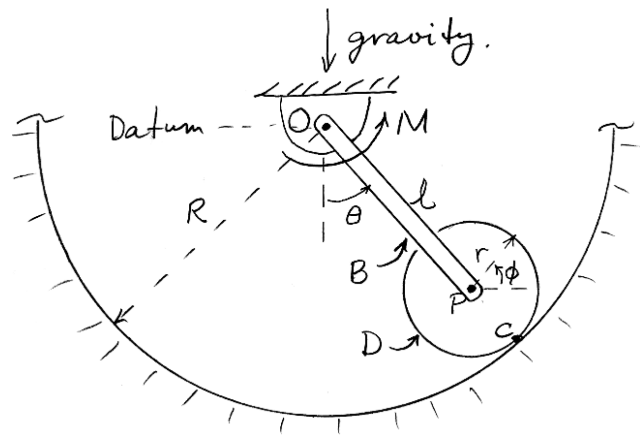
ME 555 Intermediate Dynamics
Lagrange's Equations Examples

Example #1

The system at the right consists of two bodies, a slender bar *B* and a disk *D*, moving together in a vertical plane. As *B* rotates about *O*, *D* rolls without slipping on the fixed circular outer surface. The length of *B* is ℓ, the radius of *D* is r, and the radius of the outer surface is R. The mass of the bar and disk are both m. The system is driven by the torque $M(t)$.



Equation of Motion

Using θ as the single generalized coordinate, the equation of motion of the system may be found from Lagrange's equation

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = F_\theta} \qquad (1.1)$$

where

$$L = K - V = K_B + K_D - V_B - V_D$$

$$K_D = \tfrac{1}{2}\underset{\sim}{\omega}_D \cdot \underset{\sim}{H}_C = \tfrac{1}{2} I_C \dot{\phi}^2 \qquad \text{(fixed axis rotation)}$$
$$= \tfrac{1}{2}(\tfrac{1}{2}mr^2 + mr^2)\dot{\phi}^2$$
$$= \boxed{\tfrac{3}{4}mr^2\dot{\phi}^2}$$

$$K_B = \tfrac{1}{2}\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_O = \tfrac{1}{2} I_O \dot{\theta}^2 \qquad \text{(fixed axis rotation)}$$
$$= \tfrac{1}{2}(\tfrac{1}{3}m\ell^2)\dot{\theta}^2$$
$$= \boxed{\tfrac{1}{6}m\ell^2\dot{\theta}^2}$$

$$V = V_D + V_B = -mg\ell C_\theta - \tfrac{1}{2}mg\ell C_\theta = \boxed{-\tfrac{3}{2}mg\ell C_\theta}$$

To express L in terms of θ and $\dot{\theta}$ only, we can use the concept of instantaneous centers to write $\boxed{v_P = \ell\dot{\theta} = -r\dot{\phi}}$. Using this equation to remove $\dot{\phi}$ from the Lagrangian gives

$$\boxed{L = \tfrac{11}{12} m\ell^2\dot{\theta}^2 + \tfrac{3}{2} mg\ell C_\theta}$$

The generalized active force F_θ and the derivatives of the Lagrangian can then be calculated as

$$F_\theta = M\,\underset{\sim}{k}\cdot\frac{\partial}{\partial\dot{\theta}}\left(\underset{\sim}{\omega}_B\right) = M\,\underset{\sim}{k}\cdot\underset{\sim}{k} = M(t)$$

$$\frac{\partial L}{\partial\dot{\theta}} = \tfrac{11}{6} m\ell^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \tfrac{11}{6} m\ell^2\ddot{\theta}$$

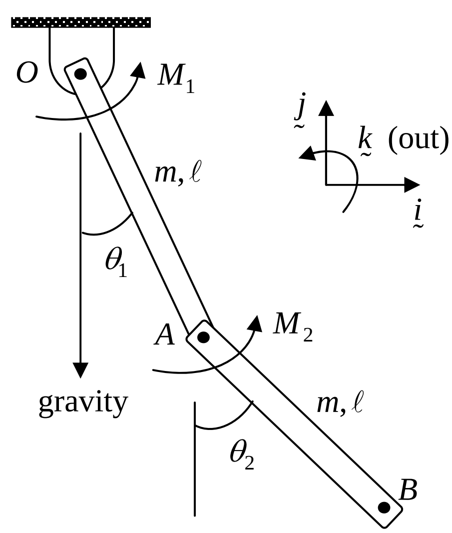
$$\frac{\partial L}{\partial\theta} = -\tfrac{3}{2} mg\ell S_\theta$$

Substituting into Lagrange's equation (1.1) gives the equation of motion

$$\boxed{\tfrac{11}{6} m\ell^2\ddot{\theta} + \tfrac{3}{2} mg\ell S_\theta = M(t)}$$

Example #2 – Double Pendulum

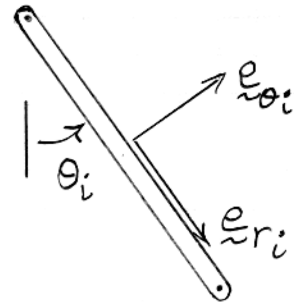
The figure to the right shows a double pendulum in a vertical plane with driving torques at the joints. The two uniform slender links are assumed to be identical with mass m and length ℓ. The system has two degrees of freedom described by the generalized coordinate set (θ_1, θ_2).



Equation of Motion

Using θ_1 and θ_2 as the two generalized coordinates, the equations of motion of the system may be found from Lagrange's equations

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= F_{\theta_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= F_{\theta_2} \end{aligned}}$$



Typical Link

Kinematics

Using the concept of relative velocity, the velocities and squares of velocities of the mass centers of the two links may be written as

$$\begin{aligned}
\underset{\sim}{v}_{G_1} &= \underset{\sim}{v}_O + \underset{\sim}{v}_{G_1/O} = \tfrac{1}{2}\ell\dot{\theta}_1\,\underset{\sim}{e}_{\theta_1} \\
\underset{\sim}{v}_{G_2} &= \underset{\sim}{v}_A + \underset{\sim}{v}_{G_2/A} = \underset{\sim}{v}_O + \underset{\sim}{v}_{A/O} + \underset{\sim}{v}_{G_2/A} = \ell\dot{\theta}_1\,\underset{\sim}{e}_{\theta_1} + \tfrac{1}{2}\ell\dot{\theta}_2\,\underset{\sim}{e}_{\theta_2} \\
v_{G_1}^2 &= \underset{\sim}{v}_{G_1}\cdot\underset{\sim}{v}_{G_1} = \tfrac{1}{4}\ell^2\dot{\theta}_1^2 \\
v_{G_2}^2 &= \underset{\sim}{v}_{G_2}\cdot\underset{\sim}{v}_{G_2} = \ell^2\dot{\theta}_1^2 + \tfrac{1}{4}\ell^2\dot{\theta}_2^2 + 2\left(\tfrac{1}{2}\ell^2\dot{\theta}_1\dot{\theta}_2\right)\left(\underset{\sim}{e}_{\theta_1}\cdot\underset{\sim}{e}_{\theta_2}\right) = \ell^2\dot{\theta}_1^2 + \tfrac{1}{4}\ell^2\dot{\theta}_2^2 + \ell^2\dot{\theta}_1\dot{\theta}_2 C_{2-1}
\end{aligned}$$

Kinetic Energy

The kinetic energy of the system may then be written

$$K = K_1 + K_2$$

where

$$K_1 = \tfrac{1}{2}I_O\dot{\theta}_1^2 = \tfrac{1}{2}\left(\tfrac{1}{3}m\ell^2\right)\dot{\theta}_1^2 = \tfrac{1}{6}m\ell^2\dot{\theta}_1^2 \qquad \text{(fixed axis rotation)}$$

$$\begin{aligned}
K_2 &= \tfrac{1}{2}mv_{G_2}^2 + \tfrac{1}{2}I_{G_2}\dot{\theta}_2^2 \qquad \text{(general plane motion)} \\
&= \tfrac{1}{2}m\ell^2\dot{\theta}_1^2 + \tfrac{1}{8}m\ell^2\dot{\theta}_2^2 + \tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 C_{2-1} + \tfrac{1}{24}m\ell^2\dot{\theta}_2^2 \\
&= \tfrac{1}{2}m\ell^2\dot{\theta}_1^2 + \tfrac{1}{6}m\ell^2\dot{\theta}_2^2 + \tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 C_{2-1}
\end{aligned}$$

Potential Energy

Assuming the datum is level with the point *O*, the potential energy of the system can be written

$$V = V_1 + V_2 = -\tfrac{1}{2}mg\ell C_1 - mg\left(\ell C_1 + \tfrac{1}{2}\ell C_2\right) = -\tfrac{3}{2}mg\ell C_1 - \tfrac{1}{2}mg\ell C_2$$

Lagrangian $L = K - V$

$$\boxed{L = \tfrac{2}{3}m\ell^2\dot{\theta}_1^2 + \tfrac{1}{6}m\ell^2\dot{\theta}_2^2 + \tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 C_{2-1} + \tfrac{3}{2}mg\ell C_1 + \tfrac{1}{2}mg\ell C_2}$$

Generalized Forces

The generalized forces associated with the driving torques are

$$F_{\theta_1} = \left(M_1 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(-M_2 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(M_2 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_2}{\partial \dot{\theta}_1} \right) = \boxed{M_1 - M_2}$$

$$F_{\theta_2} = \left(M_1 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(-M_2 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(M_2 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_2}{\partial \dot{\theta}_2} \right) = \boxed{M_2}$$

Derivatives of Lagrangian

$$\frac{\partial L}{\partial \dot{\theta}_1} = \tfrac{4}{3}m\ell^2\dot{\theta}_1 + \tfrac{1}{2}m\ell^2\dot{\theta}_2 C_{2-1}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \tfrac{4}{3}m\ell^2\ddot{\theta}_1 + \tfrac{1}{2}m\ell^2 C_{2-1}\ddot{\theta}_2 - \tfrac{1}{2}m\ell^2\dot{\theta}_2\left(\dot{\theta}_2 - \dot{\theta}_1\right)S_{2-1}}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \tfrac{1}{2}m\ell^2 C_{2-1}\dot{\theta}_1 + \tfrac{1}{3}m\ell^2\dot{\theta}_2$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \tfrac{1}{2}m\ell^2 C_{2-1}\ddot{\theta}_1 + \tfrac{1}{3}m\ell^2\ddot{\theta}_2 - \tfrac{1}{2}m\ell^2\dot{\theta}_1\left(\dot{\theta}_2 - \dot{\theta}_1\right)S_{2-1}}$$

$$\boxed{\frac{\partial L}{\partial \theta_1} = \tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 S_{2-1} - \tfrac{3}{2}mg\ell S_1} \qquad \boxed{\frac{\partial L}{\partial \theta_2} = -\tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 S_{2-1} - \tfrac{1}{2}mg\ell S_2}$$

Substituting into Lagrange's equations gives the following equations of motion

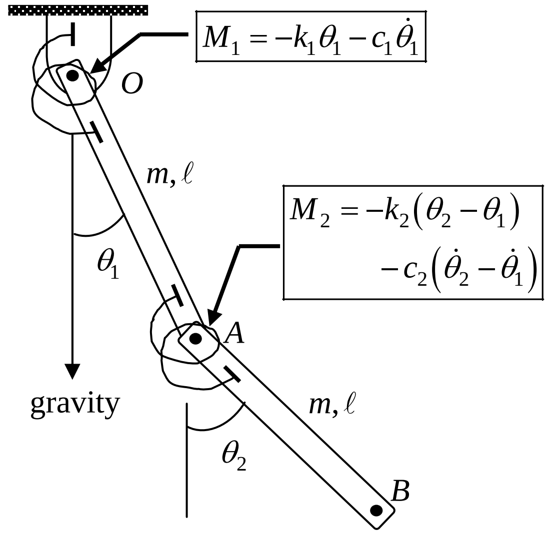
$$\boxed{\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\ddot{\theta}_2 - \left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 + \tfrac{3}{2}mg\ell S_1 = M_1(t) - M_2(t)} \qquad (1.2)$$

$$\boxed{\left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\ddot{\theta}_1 + \left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 + \tfrac{1}{2}mg\ell S_2 = M_2(t)} \qquad (1.3)$$

This is a *coupled* set of nonlinear differential equations of motion for the double pendulum.

Example – Double Pendulum with Springs and Dampers

The figure at the right shows a double pendulum as in the above example with the driving torques replaced with a set of springs and dampers. The equations of motion of this system is easily derived using the results from the previous example given that



$$M_1 = -k_1\theta_1 - c_1\dot{\theta}_1$$
$$M_2 = -k_2(\theta_2 - \theta_1) - c_2(\dot{\theta}_2 - \dot{\theta}_1)$$

Substituting these results into the equations (1.2) and (1.3) gives

$$\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2 - \left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2 + \tfrac{3}{2}mg\ell S_1$$
$$= -k_1\theta_1 - c_1\dot{\theta}_1 + k_2(\theta_2 - \theta_1) + c_2(\dot{\theta}_2 - \dot{\theta}_1)$$
$$\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1 + \left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2 + \tfrac{1}{2}mg\ell S_2 = -k_2(\theta_2 - \theta_1) - c_2(\dot{\theta}_2 - \dot{\theta}_1)$$

or

$$\boxed{\begin{aligned}&\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2 - \left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2 + \tfrac{3}{2}mg\ell S_1\\ &\qquad + (c_1 + c_2)\dot{\theta}_1 - c_2\dot{\theta}_2 + (k_1 + k_2)\theta_1 - k_2\theta_2 = 0\end{aligned}} \qquad (1.4)$$

$$\boxed{\begin{aligned}&\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1 + \left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2 + \tfrac{1}{2}mg\ell S_2\\ &\qquad + c_2(\dot{\theta}_2 - \dot{\theta}_1) + k_2(\theta_2 - \theta_1) = 0\end{aligned}} \qquad (1.5)$$

This is a set of *two simultaneous nonlinear differential equations of motion* of the double pendulum with springs and dampers at the connecting joints.