بسم الله الرحمن الرحیم

Mechanical Vibration

ارتعاشات مکانیکی(درس بیست و دوم)

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معادله های لاگرانژ

❖ معادله های لاگرانژ:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q_i$$

q_i : مختصه های عمومی سیستم

T : انرژی جنبشی سیستم

V : انرژی پتانسیل سیستم

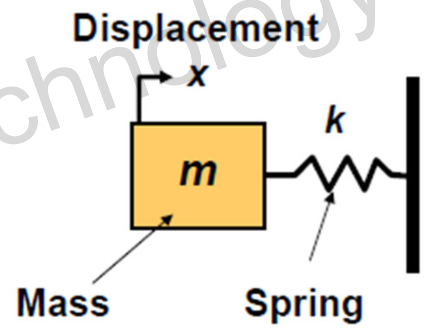
نیروهای عمومی نظیر نیروی استهلاک (نیروهایی که نمی توان آنها را از تابع پتانسیل به دست آورد):

$$Q_i=\frac{\delta W}{\delta q_i}$$

$$L=(T-U) \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0$$

معادله های لاگرانژ

❖ کاربرد معادله های لاگرانژ در استخراج معادله های حرکت را با چند مثال توضیح می دهیم:



$$T = \frac{1}{2} m\dot{x}^2, \qquad U = \frac{1}{2} kx^2$$

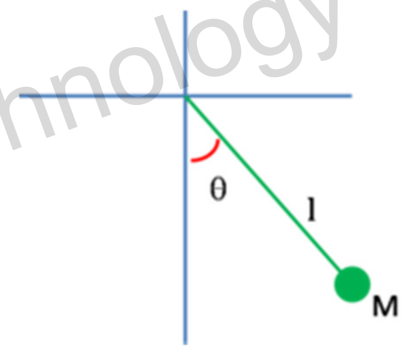
$$L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + kx = 0$$

❖ استخراج معادله حرکت پاندول ساده:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q_i$$



$$T=\frac{1}{2}M(l\dot{\theta})^{2}$$

$$V=Mgl(1-cos\theta)$$

$$\frac{\partial T}{\partial \dot{\theta}}=Ml^2\dot{\theta} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)=Ml^2\ddot{\theta}$$

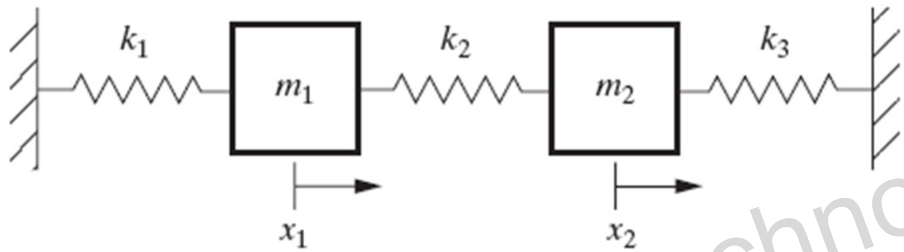
$$\frac{\partial T}{\partial \theta}=0 \qquad \frac{\partial V}{\partial \theta}=Mglsin\theta$$

$$Q_i=0$$

$$\Rightarrow \quad Ml^2\ddot{\theta}+Mglsin\theta=0$$

معادله های لاگرانژ

❖ مثال:



$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q_i$$

$$T=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2 \qquad V=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2(x_2-x_1)^2+\frac{1}{2}k_3x_2^2$$

$$\frac{\partial T}{\partial \dot{x}_1}=m_1\dot{x}_1 \;\Rightarrow\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right)=m_1\ddot{x}_1$$

$$\frac{\partial T}{\partial x_1}=0$$

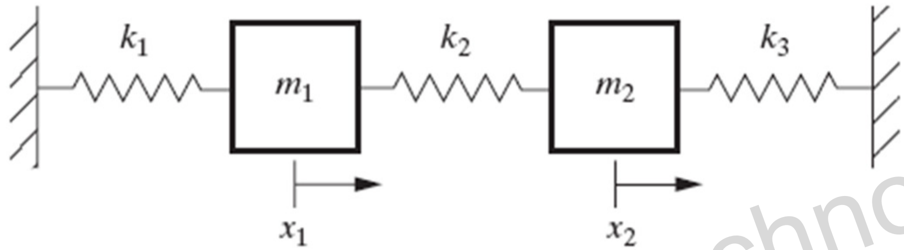
$$\frac{\partial V}{\partial x_1}=k_1x_1-k_2(x_2-x_1) \qquad Q_i=0$$

$$\Rightarrow\; m_1\ddot{x}_1+k_1x_1-k_2(x_2-x_1)=0$$

$$m_1\ddot{x}_1+(k_1+k_2)x_1-k_2x_2=0$$

معادله های لاگرانژ

❖ مثال:



$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q_i$$

$$T=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2 \qquad V=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2(x_2-x_1)^2+\frac{1}{2}k_3x_2^2$$

$$\frac{\partial T}{\partial \dot{x}_2}=m_2\dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right)=m_2\ddot{x}_2$$

$$\frac{\partial T}{\partial x_2}=0$$

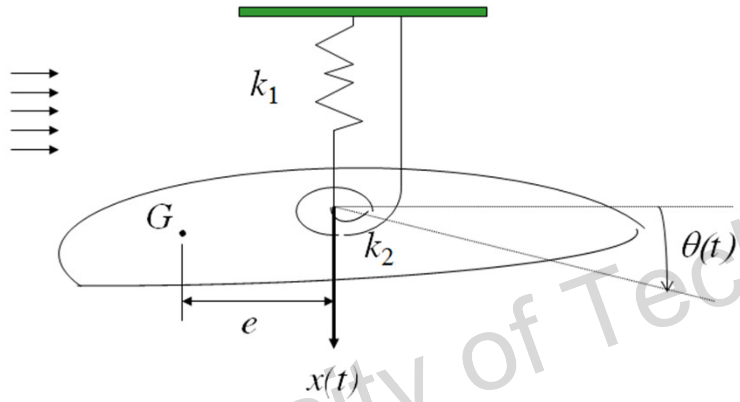
$$\frac{\partial V}{\partial x_2}=k_3x_2+k_2(x_2-x_1) \qquad Q_i=0$$

$$\Rightarrow m_2\ddot{x}_2+k_3x_2+k_2(x_2-x_1)=0$$

$$m_1\ddot{x}_1-k_2x_1+(k_2+k_3)x_2=0$$

معادله های لاگرانژ

❖ مثال:



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0$$

$$T=\frac{1}{2}m\dot{x}_G^2+\frac{1}{2}J\dot{\theta}^2$$

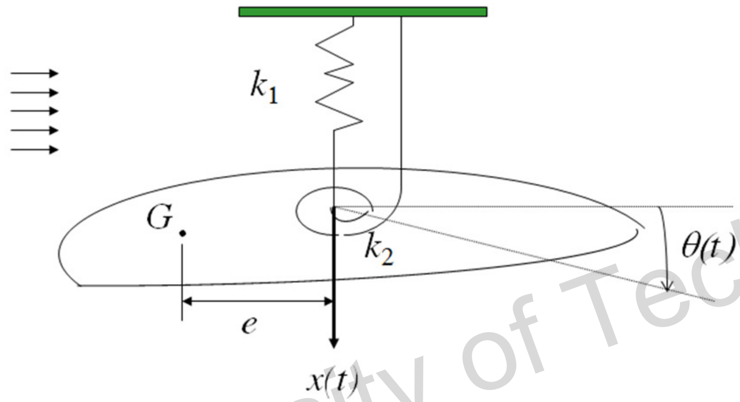
$$x_G(t)=x(t)-e\sin\theta(t)$$

$$\Rightarrow \dot{x}_G(t)=\dot{x}(t)-e\cos\theta(t)\frac{d\theta}{dt}=\dot{x}(t)-e\dot{\theta}\cos\theta(t)$$

$$T=\frac{1}{2}m[\dot{x}-e\dot{\theta}\cos\theta]^2+\frac{1}{2}J\dot{\theta}^2$$

معادله های لاگرانژ

❖ مثال:



$$T=\frac{1}{2}m[\dot{x}-e\dot{\theta}\cos\theta]^2+\frac{1}{2}J\dot{\theta}^2$$

$$U=\frac{1}{2}k_1x^2+\frac{1}{2}k_2\theta^2$$

$$L=T-U=\frac{1}{2}m\left[\dot{x}-e\dot{\theta}\cos\theta\right]^2+\frac{1}{2}J\dot{\theta}^2-\frac{1}{2}k_1x^2-\frac{1}{2}k_2\theta^2$$

معادله های لاگرانژ

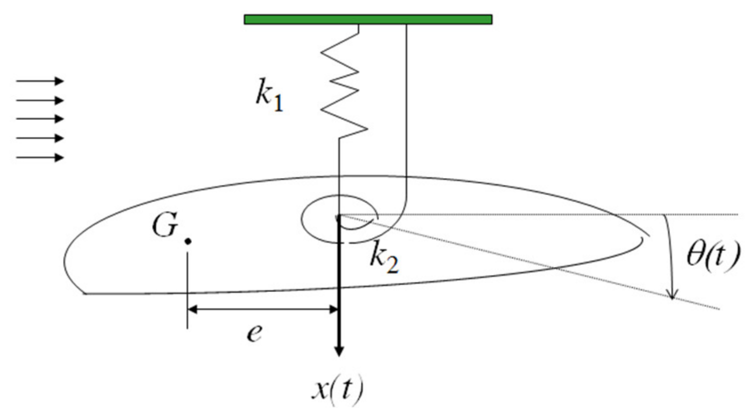
❖ مثال:

$$L = T - U = \frac{1}{2} m\left[\dot{x} - e\dot{\theta}\cos\theta \right]^2 + \frac{1}{2} J\dot{\theta}^2 - \frac{1}{2} k_1 x^2 - \frac{1}{2} k_2 \theta^2$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{x}} = m[\dot{x} - e\dot{\theta}c\,\mathrm{os}\,\theta]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = m\ddot{x} - me\ddot{\theta} + me\dot{\theta}^2 \sin\theta$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial x} = -k_1 x$$



- مثال:

$$\frac{\partial L}{\partial \dot{q}_1}=\frac{\partial L}{\partial \dot{x}}=m[\dot{x}-e\dot{\theta}c\,\mathrm{os}\,\theta]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=m\ddot{x}-me\ddot{\theta}+me\dot{\theta}^2\sin\theta$$

$$\frac{\partial L}{\partial q_1}=\frac{\partial L}{\partial x}=-k_1x$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0$$

$$m\ddot{x}-me\ddot{\theta}\cos\theta+em\dot{\theta}^2\sin\theta+k_1x=0$$

$$J\ddot{\theta}+me\cos\theta\ddot{x}+me^2\cos^2\theta\ddot{\theta}-me^2\dot{\theta}^2\sin\theta\cos\theta+k_2\theta=0$$

❖ مثال:

$$m\ddot{x} - me\ddot{\theta}\cos\theta + em\dot{\theta}^2\sin\theta + k_1 x = 0$$

$$J\ddot{\theta} + me\cos\theta\ddot{x} + me^2\cos^2\theta\ddot{\theta} - me^2\dot{\theta}^2\sin\theta\cos\theta + k_2\theta = 0$$

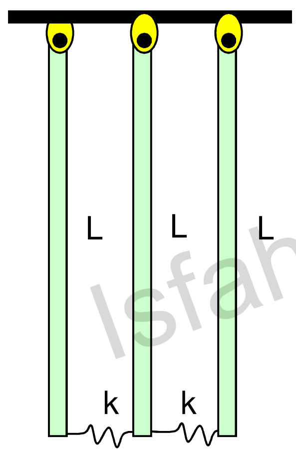
small angle approximations: $\sin\theta \rightarrow \theta \quad \cos\theta \rightarrow 1$

$$\begin{bmatrix} m & -me \\ -me & me^2 + J \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

سیستم های چند درجه آزادی

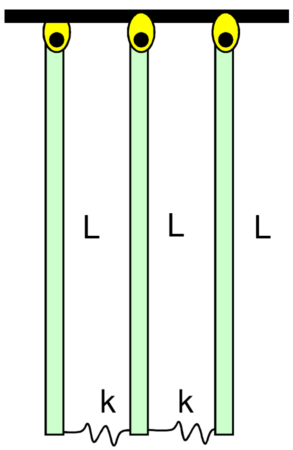
❖ مثال:

سیستم سه درجه آزادی شکل زیر را در نظر بگیرید. در حالت اولیه، هیچیک از فنرها کشیدگی یا فشردگی ندارند. معادلات حرکت سیستم را با استفاده از روش لاگرانژ بنویسید و ماتریس‌های جرم و سختی سیستم را به دست آورید.



طول هر میله: $L = 1$ m؛ جرم هر میله: $m = 1$ kg؛ سختی فنرها: $K = 100$ N/m؛ ثابت جاذبه: $g = 10$ m/s.

سیستم های چند درجه آزادی



$$T=\frac{1}{2}I_O\dot{\theta}_1^2+\frac{1}{2}I_O\dot{\theta}_2^2+\frac{1}{2}I_O\dot{\theta}_3^2$$

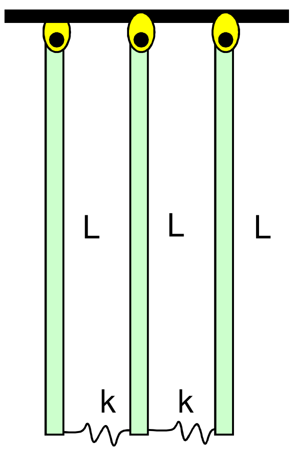
$$V_e=\frac{1}{2}kL^2(\theta_2-\theta_1)^2+\frac{1}{2}kL^2(\theta_3-\theta_2)^2$$

$$V_g=mg\frac{L}{2}(1-\cos\theta_1)+mg\frac{L}{2}(1-\cos\theta_2)+mg\frac{L}{2}(1-\cos\theta_3)$$

V_e انرژی پتانسیل ناشی از فنرها و V_g انرژی پتانسیل ناشی از ثقل یا وزن میله‌هاست.

$$V=\frac{1}{2}kL^2(\theta_2-\theta_1)^2+\frac{1}{2}kL^2(\theta_3-\theta_2)^2+mg\frac{L}{2}(3-\cos\theta_1-\cos\theta_2-\cos\theta_3)$$

سیستم های چند درجه آزادی



$$T = \frac{1}{2} I_O \dot{\theta}_1^2 + \frac{1}{2} I_O \dot{\theta}_2^2 + \frac{1}{2} I_O \dot{\theta}_3^2$$

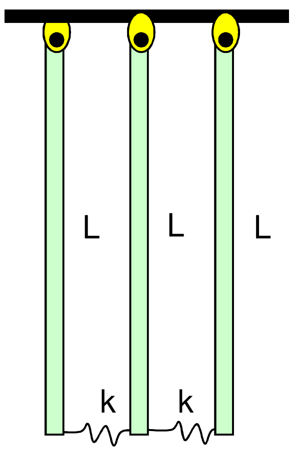
$$V = \frac{1}{2} kL^2 (\theta_2 - \theta_1)^2 + \frac{1}{2} kL^2 (\theta_3 - \theta_2)^2 + mg\frac{L}{2}(3 - \cos\theta_1 - \cos\theta_2 - \cos\theta_3)$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = I_O \dot{\theta}_2 \quad \rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = I_O \ddot{\theta}_2$$

$$\frac{\partial V}{\partial \theta_2} = kL^2 (\theta_2 - \theta_1) - kL^2 (\theta_3 - \theta_2) + mg\frac{L}{2}\sin\theta_2$$

$$I_O \ddot{\theta}_2 - kL^2 \theta_1 + 2kL^2 \theta_2 - kL^2 \theta_3 + mg\frac{L}{2}\theta_2 = 0$$

سیستم های چند درجه آزادی



$$T=\frac{1}{2}I_O\dot{\theta}_1^2+\frac{1}{2}I_O\dot{\theta}_2^2+\frac{1}{2}I_O\dot{\theta}_3^2$$

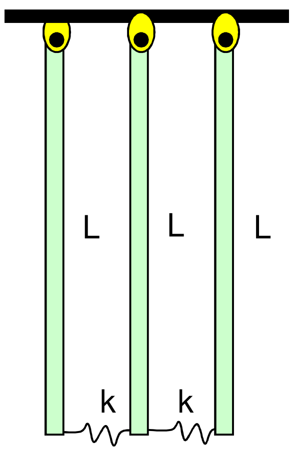
$$V=\frac{1}{2}kL^2(\theta_2-\theta_1)^2+\frac{1}{2}kL^2(\theta_3-\theta_2)^2+mg\frac{L}{2}(3-\cos\theta_1-\cos\theta_2-\cos\theta_3)$$

$$\frac{\partial T}{\partial\dot{\theta}_3}=I_O\dot{\theta}_3 \rightarrow \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial\dot{\theta}_3}\right)=I_O\ddot{\theta}_3$$

$$\frac{\partial V}{\partial\theta_3}=kL^2(\theta_3-\theta_2)+mg\frac{L}{2}\sin\theta_3$$

$$I_O\ddot{\theta}_3+kL^2(\theta_3-\theta_2)+mg\frac{L}{2}\theta_3=0$$

سیستم های چند درجه آزادی

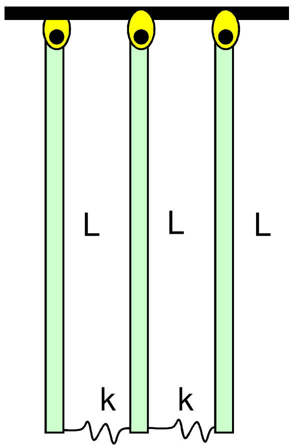


$$\begin{cases} \frac{1}{3}mL^2\ddot{\theta}_1 + kL^2\theta_1 - kL^2\theta_2 + mg\frac{L}{2}\theta_1 = 0 \\ \frac{1}{3}mL^2\ddot{\theta}_2 - kL^2\theta_1 + 2kL^2\theta_2 - kL^2\theta_3 + mg\frac{L}{2}\theta_2 = 0 \\ \frac{1}{3}mL^2\ddot{\theta}_3 + kL^2\theta_3 - kL^2\theta_2 + mg\frac{L}{2}\theta_3 = 0 \end{cases}$$

$$\begin{bmatrix} I_O & 0 & 0 \\ 0 & I_O & 0 \\ 0 & 0 & I_O \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + L^2 \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} + \begin{bmatrix} mgL/2 & 0 & 0 \\ 0 & mgL/2 & 0 \\ 0 & 0 & mgL/2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} 100+5 & -100 & 0 \\ -100 & 200+5 & -100 \\ 0 & -100 & 100+5 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

سیستم های چند درجه آزادی



$$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} 100+5 & -100 & 0 \\ -100 & 200+5 & -100 \\ 0 & -100 & 100+5 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\omega_1 = \sqrt{15} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\omega_2 = \sqrt{315} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

$$\omega_3 = \sqrt{915} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$