10.

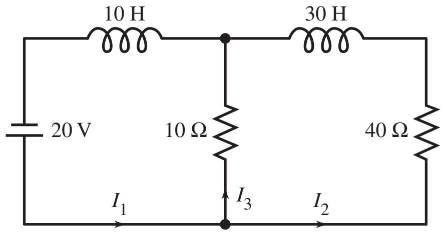


Figure 5.39 *RL* network for Problem 10

11.

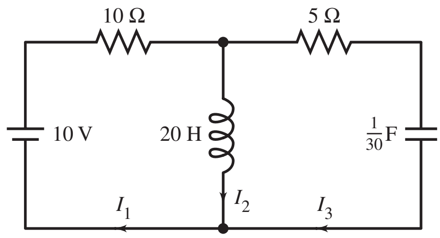


Figure 5.40 *RLC* network for Problem 11

12.

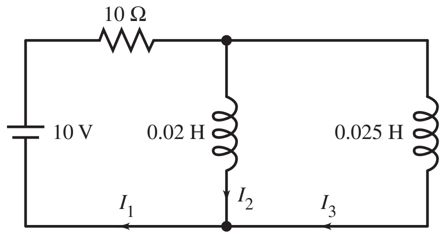


Figure 5.41 *RL* network for Problem 12

13.

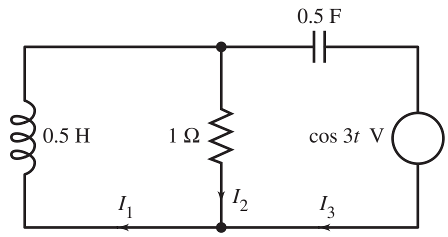


Figure 5.42 *RLC* network for Problem 13

5.8 DYNAMICAL SYSTEMS, POINCARÉ MAPS, AND CHAOS

In this section we take an excursion through an area of mathematics that has received a lot of attention both for the interesting mathematical phenomena being observed and for its application to fields such as meteorology, heat conduction, fluid mechanics, lasers, chemical reactions, and nonlinear circuits, among others. The area is that of nonlinear dynamical systems.[†]

A **dynamical system** is any system that allows one to determine (at least theoretically) the future states of the system given its present or past state. For example, the recursive formula (difference equation)

$$x_{n+1} = (1.05)x_n , \qquad n = 0, 1, 2, \ldots$$

is a dynamical system, since we can determine the next state, x_{n+1}, given the previous state, x_n. If we know x_0, then we can compute any future state $\left[\text{indeed, } x_{n+1} = x_0(1.05)^{n+1}\right]$.

Another example of a dynamical system is provided by the differential equation

$$\frac{dx}{dt} = -2x ,$$

where the solution $x(t)$ specifies the state of the system at "time" t. If we know $x(t_0) = x_0$, then we can determine the state of the system at any future time $t > t_0$ by solving the initial value problem

$$\frac{dx}{dt} = -2x , \qquad x(t_0) = x_0 .$$

Indeed, a simple calculation yields $x(t) = x_0 e^{-2(t-t_0)}$ for $t \geq t_0$.

[†]For a more detailed study of dynamical systems, see *An Introduction to Chaotic Dynamical Systems,* 2nd ed., by R. L. Devaney (Addison-Wesley, Reading, Mass., 1989) and *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields,* by J. Guckenheimer and P. J. Holmes (Springer-Verlag, New York, 1983).

For a dynamical system defined by a differential equation, it is often helpful to work with a related dynamical system defined by a difference equation. For example, when we cannot express the solution to a differential equation using elementary functions, we can use a numerical technique such as the improved Euler's method or Runge–Kutta to approximate the solution to an initial value problem. This numerical scheme defines a new (but related) dynamical system that is often easier to study.

In Section 5.4, we used phase plane diagrams to study autonomous systems in the plane. Many important features of the system can be detected just by looking at these diagrams. For example, a closed trajectory corresponds to a periodic solution. The trajectories for *non*autonomous systems in the phase plane are much more complicated to decipher. One technique that is helpful in this regard is the so-called **Poincaré map.** As we will see, these maps replace the study of a nonautonomous system with the study of a dynamical system defined by the location in the xv-plane $(v = dx/dt)$ of the solution at regularly spaced moments in time such as $t = 2\pi n$, where $n = 0, 1, 2, \ldots$. The advantage in using the Poincaré map will become clear when the method is applied to a nonlinear problem for which no explicit solution is known. In such a case, the trajectories are computed using a numerical scheme such as Runge–Kutta. Several software packages have options that will construct Poincaré maps for a given system.

To illustrate the Poincaré map, consider the equation

(1) $$x''(t) + \omega^2 x(t) = F\cos t\ ,$$

where F and ω are positive constants. We studied similar equations in Section 4.10 and found that a general solution for $\omega \neq 1$ is given by

(2) $$x(t) = A\sin(\omega t + \phi) + \frac{F}{\omega^2 - 1}\cos t\ ,$$

where the amplitude A and the phase angle ϕ are arbitrary constants. Since $v = x'$,

$$v(t) = \omega A\cos(\omega t + \phi) - \frac{F}{\omega^2 - 1}\sin t\ .$$

Because the forcing function $F\cos t$ is 2π-periodic, it is natural to seek 2π-periodic solutions to (1). For this purpose, we define the Poincaré map as

(3) $$\begin{aligned} x_n &:= x(2\pi n) = A\sin(2\pi\omega n + \phi) + F/(\omega^2 - 1)\ , \\ v_n &:= v(2\pi n) = \omega A\cos(2\pi\omega n + \phi)\ , \end{aligned}$$

for $n = 0, 1, 2, \ldots$. In Figure 5.43 on page 299, we plotted the first 100 values of (x_n, v_n) in the xv-plane for different choices of ω. For simplicity, we have taken $A = F = 1$ and $\phi = 0$. These graphs are called **Poincaré sections.** We will interpret them shortly.

Now let's play the following game. We agree to ignore the fact that we already know the formula for $x(t)$ for all $t \geq 0$. We want to see what information about the solution we can glean just from the Poincaré section and the form of the differential equation.

Notice that the first two Poincaré sections in Figure 5.43, corresponding to $\omega = 2$ and 3, consist of a single point. This tells us that, starting with $t = 0$, every increment 2π of t returns us to the same point in the phase plane. This in turn implies that equation (1) has a 2π-periodic solution, which can be proved as follows: For $\omega = 2$, let $x(t)$ be the solution to (1) with $\big(x(0), v(0)\big) = (1/3, 2)$ and let $X(t) := x(t + 2\pi)$. Since the Poincaré section is just the point $(1/3, 2)$, we have $X(0) = x(2\pi) = 1/3$ and $X'(0) = x'(2\pi) = 2$. Thus, $x(t)$ and $X(t)$ have the same initial values at $t = 0$. Further, because $\cos t$ is 2π-periodic, we also have

$$X''(t) + \omega^2 X(t) = x''(t + 2\pi) + \omega^2 x(t + 2\pi) = \cos(t + 2\pi) = \cos t\ .$$

Consequently, $x(t)$ and $X(t)$ satisfy the same initial value problem. By the uniqueness theorems of Sections 4.2 and 4.5, these functions must agree on the interval $[0, \infty)$. Hence, $x(t) = X(t) = x(t + 2\pi)$ for all $t \geq 0$; that is, $x(t)$ is 2π-periodic. (The same reasoning works for $\omega = 3$.) With a similar argument, it follows from the Poincaré section for $\omega = 1/2$ that there is a solution of period 4π that alternates between the two points displayed in Figure 5.43(c) as t is incremented by 2π. For the case $\omega = 1/3$, we deduce that there is a solution of period 6π rotating among three points, and for $\omega = 1/4$, there is an 8π-periodic solution rotating among four points. We call these last three solutions **subharmonics.**

The case $\omega = \sqrt{2}$ is different. So far, in Figure 5.43(f), none of the points has repeated. Did we stop too soon? Will the points ever repeat? Here, the fact that $\sqrt{2}$ is irrational plays a crucial role. It turns out that every integer n yields a distinct point in the Poincaré section (see Problem 8). However, there is a pattern developing. The points all appear to lie on a simple curve, possibly an ellipse. To see that this is indeed the case, notice that when $\omega = \sqrt{2}$, $A = F = 1$, and $\phi = 0$, we have

$$x_n = \sin\left(2\sqrt{2}\pi n\right) + 1 \ , \qquad v_n = \sqrt{2}\cos\left(2\sqrt{2}\pi n\right) \ , \qquad n = 0, 1, 2, \ldots .$$

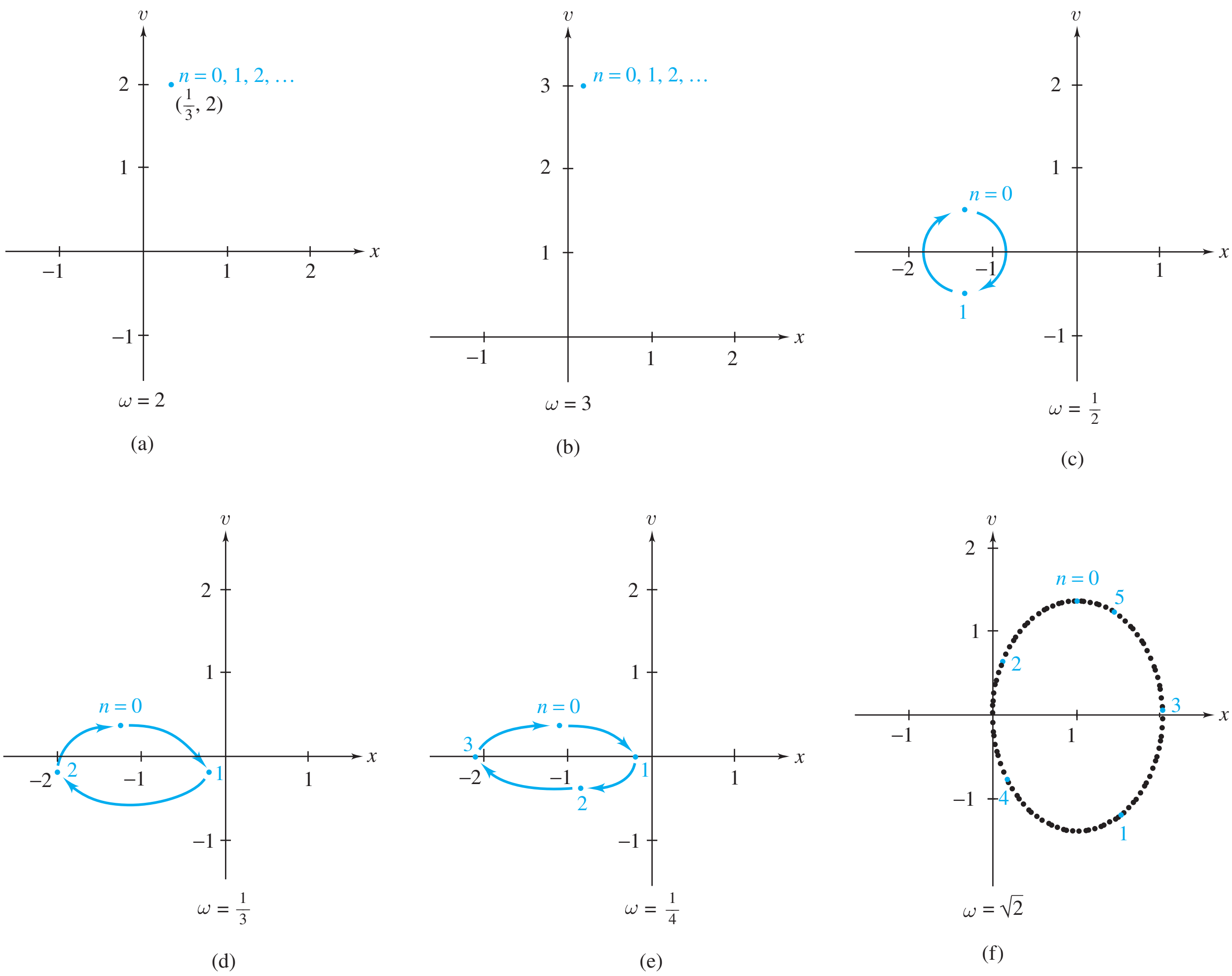


Figure 5.43 Poincaré sections for equation (1) for various values of ω

It is then an easy computation to show that each (x_n, v_n) lies on the ellipse

$$(x-1)^2 + \frac{v^2}{2} = 1 \ .$$

In our investigation of equation (1), we concentrated on 2π-periodic solutions because the forcing term $F\cos t$ has period 2π. [We observed subharmonics when $\omega = 1/2, 1/3$, and $1/4$—that is, solutions with periods $2(2\pi)$, $3(2\pi)$, and $4(2\pi)$.] When a damping term is introduced into the differential equation, the Poincaré map displays a different behavior. Recall that the solution will now be the sum of a transient and a steady-state term. For example, let's analyze the equation

$$x''(t) + bx'(t) + \omega^2 x(t) = F\cos t \ , \tag{4}$$

where b, F, and ω are positive constants.

When $b^2 < 4\omega^2$, the solution to (4) can be expressed as

$$x(t) = Ae^{-(b/2)t}\sin\left(\frac{\sqrt{4\omega^2 - b^2}}{2}t + \phi\right) + \frac{F}{\sqrt{(\omega^2-1)^2 + b^2}}\sin(t+\theta) \ , \tag{5}$$

where $\tan\theta = (\omega^2 - 1)/b$ and A and ϕ are arbitrary constants [compare equations (7) and (8) in Section 4.10]. The first term on the right-hand side of (5) is the transient and the second term, the steady-state solution. Let's construct the Poincaré map using $t = 2\pi n$, $n = 0, 1, 2, \ldots$. We will take $b = 0.22$, $\omega = A = F = 1$, and $\phi = 0$ to simplify the computations. Because $\tan\theta = (\omega^2 - 1)/b = 0$, we will take $\theta = 0$ as well. Then we have

$$x(2\pi n) = x_n = e^{-0.22\pi n}\sin\left(\sqrt{0.9879}\,2\pi n\right) \ ,$$

$$x'(2\pi n) = v_n = -0.11e^{-0.22\pi n}\sin\left(\sqrt{0.9879}\,2\pi n\right)$$

$$+ \sqrt{0.9879}\,e^{-0.22\pi n}\cos\left(\sqrt{0.9879}\,2\pi n\right) + \frac{1}{(0.22)} \ .$$

The Poincaré section for $n = 0, 1, 2, \ldots, 10$ is shown in Figure 5.44 (black points). After just a few iterations, we observe that $x_n \approx 0$ and $v_n \approx 1/(0.22) \approx 4.545$; that is, the points of the Poincaré section are approaching a single point in the xv-plane (colored point). Thus, we might expect that there is a 2π-periodic solution corresponding to a particular choice of A and ϕ. [In this example, where we can explicitly represent the solution, we see that indeed a 2π-periodic solution arises when we take $A = 0$ in (5).]

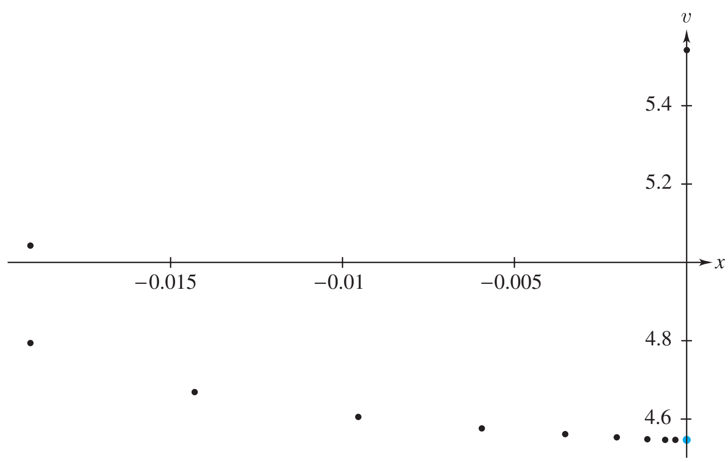


Figure 5.44 Poincaré section for equation (4) with $F = 1$, $b = 0.22$, and $\omega = 1$

There is an important difference between the Poincaré sections for equation (1) and those for equation (4). In Figure 5.43, the location of all of the points in (a)–(e) depends on the initial value selected (here $A = 1$ and $\phi = 0$). (See Problem 10.) However, in Figure 5.44, the first few points (black points) depend on the initial conditions, while the limit point (colored point) does *not* (see Problem 6). The latter behavior is typical for equations that have a "damping" term (i.e., $b > 0$); namely, the Poincaré section has a limit set† that is essentially independent of the initial conditions.

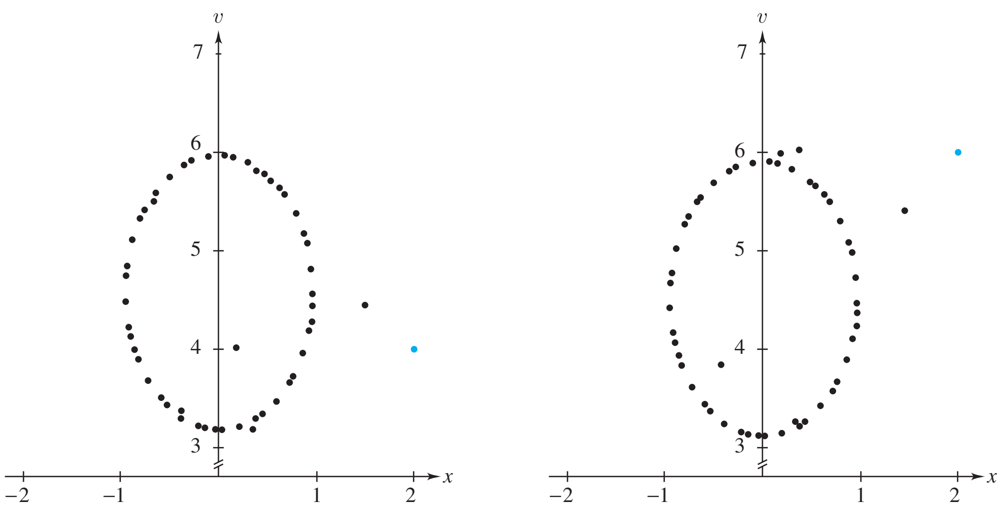
For equations with damping, the limit set may be more complicated than just a point. For example, the Poincaré map for the equation

$$x''(t) + (0.22)x'(t) + x(t) = \cos t + \cos\left(\sqrt{2}t\right) \tag{6}$$

has a limit set consisting of an ellipse (see Problem 11). This is illustrated in Figure 5.45 for the initial values $x_0 = 2$, $v_0 = 4$ and $x_0 = 2$, $v_0 = 6$.

So far we have seen limit sets for the Poincaré map that were either a single point or an ellipse—independent of the initial values. These particular limit sets are **attractors.** In general, an attractor is a set A with the property that there exists an open set†† B containing A such that whenever the Poincaré map enters B, its points remain in B *and* the limit set of the Poincaré map is a subset of A. Further, we assume A has the *invariance property:* Whenever the Poincaré map starts at a point in A, it remains in A.

In the previous examples, the attractors of the dynamical system (Poincaré map) were easy to describe. In recent years, however, many investigators, working on a variety of applications, have encountered dynamical systems that do *not* behave orderly—their attractor sets are very complicated (not just isolated points or familiar geometric objects such as ellipses). The behavior of such systems is called **chaotic,** and the corresponding limit sets are referred to as **strange attractors.**



(a) $x_0 = 2$, $v_0 = 4$ (b) $x_0 = 2$, $v_0 = 6$

Figure 5.45 Poincaré section for equation (6) with initial values x_0, v_0

†The **limit set** for a map (x_n, v_n), $n = 1, 2, 3, \ldots$, is the set of points (p, q) such that $\lim_{k\to\infty}(x_{n_k}, v_{n_k}) = (p, q)$, where $n_1 < n_2 < n_3 < \cdots$ is some subsequence of the positive integers.

††A set $B \subset \mathbf{R}^2$ is an **open set** if for each point $p \in B$ there is an open disk V containing p such that $V \subset B$.

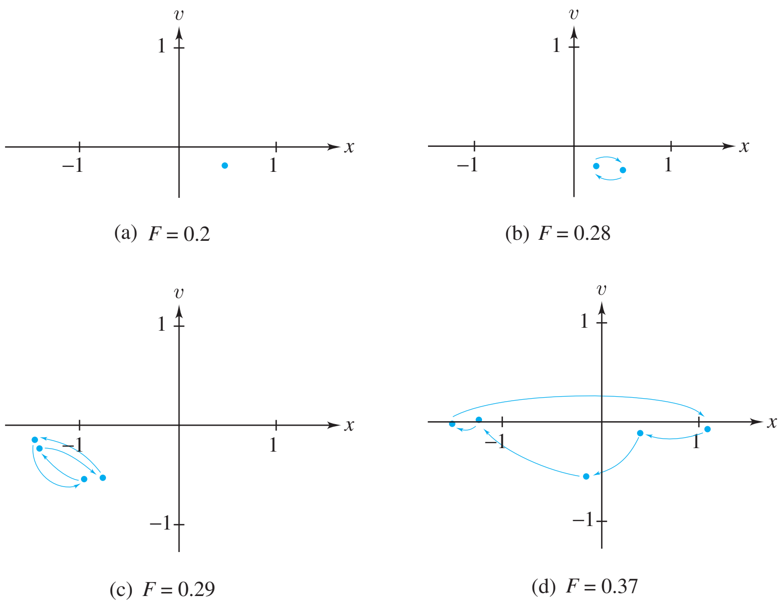


Figure 5.46 Poincaré sections for the Duffing equation (7) with $b = 0.3$ and $\gamma = 1.2$

To illustrate chaotic behavior and what is meant by a strange attractor, we discuss two nonlinear differential equations and a simple difference equation. First, let's consider the **forced Duffing equation**

$$(7) \qquad x''(t) + bx'(t) - x(t) + x^3(t) = F \sin \gamma t \ .$$

We cannot express the solution to (7) in any explicit form, so we must obtain the Poincaré map by numerically approximating the solution to (7) for fixed initial values and then plot the approximations for $x(2\pi n/\gamma)$ and $v(2\pi n/\gamma) = x'(2\pi n/\gamma)$. (Because the forcing term $F \sin \gamma t$ has period $2\pi/\gamma$, we seek $2\pi/\gamma$-periodic solutions and subharmonics.) In Figure 5.46, we display the limit sets (attractors) when $b = 0.3$ and $\gamma = 1.2$ in the cases (a) $F = 0.2$, (b) $F = 0.28$, (c) $F = 0.29$, and (d) $F = 0.37$.

Notice that as the constant F increases, the Poincaré map changes character. When $F = 0.2$, the Poincaré section tells us that there is a $2\pi/\gamma$-periodic solution. For $F = 0.28$, there is a subharmonic of period $4\pi/\gamma$, and for $F = 0.29$ and 0.37, there are subharmonics with periods $8\pi/\gamma$ and $10\pi/\gamma$, respectively.

Things are dramatically different when $F = 0.5$: The solution is neither $2\pi/\gamma$-periodic nor subharmonic. The Poincaré section for $F = 0.5$ is illustrated in Figure 5.47 on page 303. This section was generated by numerically approximating the solution to (7) when $\gamma = 1.2$, $b = 0.3$, and $F = 0.5$, for fixed initial values.† Not all of the approximations $x(2\pi n/\gamma)$ and $v(2\pi n/\gamma)$ that were calculated are graphed; because of the presence of a transient solution, the first few points were omitted. It turns out that the plotted set is essentially independent of the initial values and has the property that once a point is in the set, all subsequent points lie in the set. Because of the

†*Historical Footnote:* When researchers first encountered these strange-looking Poincaré sections, they would check their computations using different computers and different numerical schemes [see Hénon and Heiles, "The Applicability of the Third Integral of Motion: Some Numerical Experiments," *Astronomical Journal,* Vol. 69 (1964): 75]. For special types of dynamical systems, such as the Hénon map, it can be shown that there exists a true trajectory that *shadows* the numerical trajectory [see M. Hammel, J. A. Yorke, and C. Grebogi, "Numerical Orbits of Chaotic Processes Represent True Orbits," *Bulletin American Mathematical Society,* Vol. 19 (1988): 466–469].

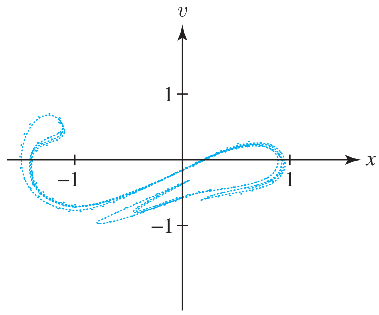


Figure 5.47 Poincaré section for the Duffing equation (7) with $b = 0.3$, $\gamma = 1.2$, and $F = 0.5$

complicated shape of the set, it is indeed a strange attractor. While the shape of the strange attractor does not depend on the initial values, the picture does change if we consider different sections; for example, $t = (2\pi n + \pi/2)/\gamma$, $n = 0, 1, 2, \ldots$ yields a different configuration.

Another example of a strange attractor occurs when we consider the **forced pendulum equation**

$$x''(t) + bx'(t) + \sin\bigl(x(t)\bigr) = F\cos t\ , \tag{8}$$

where the $x(t)$ term in (4) has been replaced by $\sin\bigl(x(t)\bigr)$. Here $x(t)$ is the angle between the pendulum and the vertical rest position, b is related to damping, and F represents the strength of the forcing function (see Figure 5.48). For $F = 2.7$ and $b = 0.22$, we have graphed in Figure 5.49 approximately 90,000 points in the Poincaré map. Since we cannot express the solution to (8) in any explicit form, the Poincaré map was obtained by numerically approximating the solution to (8) for fixed initial values and plotting the approximations for $x(2\pi n)$ and $v(2\pi n) = x'(2\pi n)$.

The Poincaré maps for the forced Duffing equation and for the forced pendulum equation not only illustrate the idea of a strange attractor; they also exhibit another peculiar behavior called **chaos.** Chaos occurs when small changes in the initial conditions lead to major changes in the behavior of the solution. Henri Poincaré described the situation as follows:

> It may happen that small differences in the initial conditions will produce very large ones in the final phenomena. A small error in the former produces an enormous error in the latter. Prediction becomes impossible . . . .

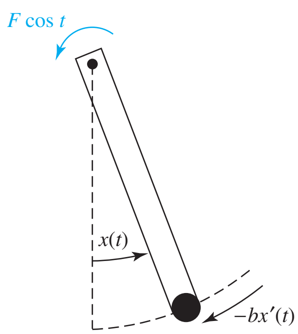


Figure 5.48 Forced damped pendulum

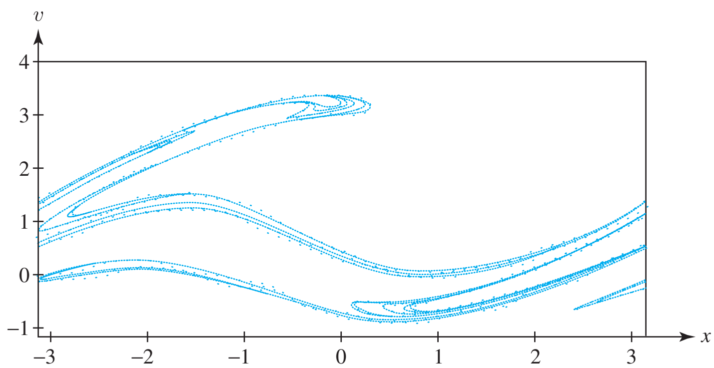


Figure 5.49 Poincaré section for the forced pendulum equation (8) with $b = 0.22$ and $F = 2.7$

In a physical experiment, we can never *exactly* (with infinite accuracy) reproduce the same initial conditions. Consequently, if the behavior is chaotic, even a slight difference in the initial conditions may lead to quite different values for the corresponding Poincaré map when n is large. Such behavior does not occur for solutions to either equation (4) or equation (1) (see Problems 6 and 7). However, two solutions to the Duffing equation (7) with $F = 0.5$ that correspond to two different but close initial values have Poincaré maps that do *not* remain close together. Although they both are attracted to the same set, their locations with respect to this set may be relatively far apart.

The phenomenon of chaos can also be illustrated by the following simple map. Let x_0 lie in $[0, 1)$ and define

(9) $$x_{n+1} = 2x_n \pmod 1 ,$$

where by (mod 1) we mean the decimal part of the number if it is greater than or equal to 1; that is

$$x_{n+1} = \begin{cases} 2x_n , & \text{for } 0 \le x_n < 1/2 , \\ 2x_n - 1 , & \text{for } 1/2 \le x_n < 1 . \end{cases}$$

When $x_0 = 1/3$, we find

$$\begin{aligned} x_1 &= 2 \cdot (1/3) \pmod 1 = 2/3 , \\ x_2 &= 2 \cdot (2/3) \pmod 1 = 4/3 \pmod 1 = 1/3 , \\ x_3 &= 2 \cdot (1/3) \pmod 1 = 2/3 , \\ x_4 &= 2 \cdot (2/3) \pmod 1 = 1/3 , \quad \text{etc.} \end{aligned}$$

Written as a sequence, we get $\{1/3, 2/3, \overline{1/3, 2/3, \ldots}\}$, where the overbar denotes the repeated pattern.

What happens when we pick a starting value x_0 near $1/3$? Does the sequence cluster about $1/3$ and $2/3$ as does the mapping when $x_0 = 1/3$? For example, when $x_0 = 0.3$, we get the sequence

$$\{0.3, 0.6, 0.2, 0.4, 0.8, \overline{0.6, 0.2, 0.4, 0.8, \ldots}\} .$$

In Figure 5.50, we have plotted the values of x_n for $x_0 = 0.3$, 0.33, and 0.333. We have not plotted the first few terms, but only those that repeat. (This omission of the first few terms parallels the situation depicted in Figure 5.47, where transient solutions arise.)

It is clear from Figure 5.50 that while the values for x_0 are getting closer to $1/3$, the corresponding maps are spreading out over the whole interval $[0, 1]$ and *not* clustering near $1/3$ and $2/3$. This behavior is chaotic, since the Poincaré maps for initial values near $1/3$ behave quite differently from the map for $x_0 = 1/3$. If we had selected x_0 to be irrational (which we can't do with a calculator), the sequence would *not* repeat and would be dense in $[0, 1]$.

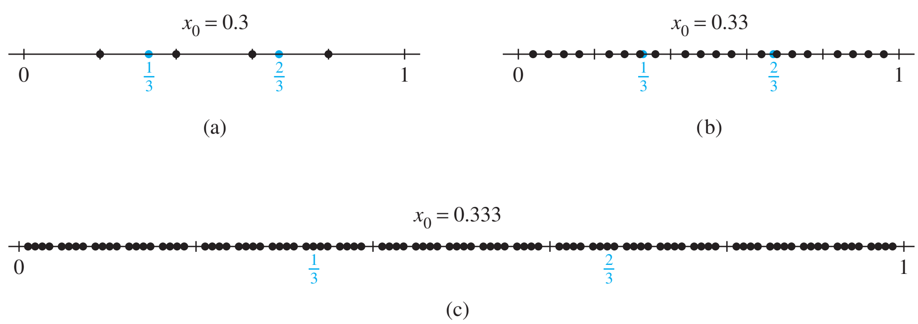


Figure 5.50 Plots of the map $x_{n+1} = 2x_n \pmod 1$ for $x_0 = 0.3$, 0.33, and 0.333

Systems that exhibit chaotic behavior arise in many applications. The challenge to engineers is to design systems that avoid this chaos and, instead, enjoy the property of **stability.** The topic of stable systems is discussed at length in Chapter 12.†

5.8 EXERCISES

A software package that supports the construction of Poincaré maps is required for the problems in this section.

1. Compute and graph the points of the Poincaré map with $t = 2\pi n$, $n = 0, 1, \ldots, 20$ for equation (1), taking $A = F = 1$, $\phi = 0$, and $\omega = 3/2$. Repeat, taking $\omega = 3/5$. Do you think the equation has a 2π-periodic solution for either choice of ω? A subharmonic solution?

2. Compute and graph the points of the Poincaré map with $t = 2\pi n$, $n = 0, 1, \ldots, 20$ for equation (1), taking $A = F = 1$, $\phi = 0$, and $\omega = 1/\sqrt{3}$. Describe the limit set for this system.

3. Compute and graph the points of the Poincaré map with $t = 2\pi n$, $n = 0, 1, \ldots, 20$ for equation (4), taking $A = F = 1$, $\phi = 0$, $\omega = 1$, and $b = -0.1$. What is happening to these points as $n \to \infty$?

4. Compute and graph the Poincaré map with $t = 2\pi n$, $n = 0, 1, \ldots, 20$ for equation (4), taking $A = F = 1$, $\phi = 0$, $\omega = 1$, and $b = 0.1$. Describe the attractor for this system.

5. Compute and graph the Poincaré map with $t = 2\pi n$, $n = 0, 1, \ldots, 20$ for equation (4), taking $A = F = 1$, $\phi = 0$, $\omega = 1/3$, and $b = 0.22$. Describe the attractor for this system.

6. Show that for $b > 0$, the Poincaré map for equation (4) is not chaotic by showing that as t gets large

$$x_n = x(2\pi n) \approx \frac{F}{\sqrt{(\omega^2 - 1)^2 + b^2}} \sin(2\pi n + \theta) \ ,$$

$$v_n = x'(2\pi n) \approx \frac{F}{\sqrt{(\omega^2 - 1)^2 + b^2}} \cos(2\pi n + \theta)$$

independent of the initial values $x_0 = x(0)$ and $v_0 = x'(0)$.

7. Show that the Poincaré map for equation (1) is not chaotic by showing that if (x_0, v_0) and (x_0^*, v_0^*) are two initial values that define the Poincaré maps $\{(x_n, v_n)\}$ and $\{(x_n^*, v_n^*)\}$, respectively, using the recursive formulas in (3), then one can make the distance between (x_n, v_n) and (x_n^*, v_n^*) small by making the distance between (x_0, v_0) and (x_0^*, v_0^*) small. [*Hint:* Let (A, ϕ) and (A^*, ϕ^*) be the polar coordinates of two points in the plane. From the law of cosines, it follows that the distance d between them is given by $d^2 = (A - A^*)^2 + 2AA^*[1 - \cos(\phi - \phi^*)]$.]

8. Consider the Poincaré maps defined in (3) with $\omega = \sqrt{2}$, $A = F = 1$, and $\phi = 0$. If this map were ever to repeat, then, for two distinct positive integers n and m, $\sin(2\sqrt{2}\pi n) = \sin(2\sqrt{2}\pi m)$. Using basic properties of the sine function, show that this would imply that $\sqrt{2}$ is rational. It follows from this contradiction that the points of the Poincaré map do not repeat.

9. The doubling modulo 1 map defined by equation (9) exhibits some fascinating behavior. Compute the sequence obtained when

(a) $x_0 = k/7$ for $k = 1, 2, \ldots, 6$.
(b) $x_0 = k/15$ for $k = 1, 2, \ldots, 14$.
(c) $x_0 = k/2^j$, where j is a positive integer and $k = 1, 2, \ldots, 2^j - 1$.

Numbers of the form $k/2^j$ are called **dyadic numbers** and are dense in $[0, 1]$. That is, there is a dyadic number arbitrarily close to any real number (rational or irrational).

10. To show that the limit set of the Poincaré map given in (3) depends on the initial values, do the following:

(a) Show that when $\omega = 2$ or 3, the Poincaré map consists of the single point

$$(x, v) = \left(A \sin\phi + \frac{F}{\omega^2 - 1} \ , \quad \omega A \cos\phi\right) .$$

†All references to Chapters 11–13 refer to the expanded text *Fundamentals of Differential Equations and Boundary Value Problems,* 6th ed.

(b) Show that when $\omega = 1/2$, the Poincaré map alternates between the two points

$$\left(\frac{F}{\omega^2 - 1} \pm A \sin \phi \ , \quad \pm \omega A \cos \phi \right) .$$

(c) Use the results of parts (a) and (b) to show that when $\omega = 2, 3$, or $1/2$, the Poincaré map (3) depends on the initial values (x_0, v_0).

11. To show that the limit set for the Poincaré map $x_n := x(2\pi n)$, $v_n := x'(2\pi n)$, where $x(t)$ is a solution to equation (6), is an ellipse and that this ellipse is the same for any initial values x_0, v_0, do the following:

(a) Argue that since the initial values affect only the transient solution to (6), the limit set for the Poincaré map is independent of the initial values.

(b) Now show that for n large,

$$x_n \approx a \sin(2\sqrt{2}\pi n + \psi) \ ,$$
$$v_n \approx c + \sqrt{2}a \cos(2\sqrt{2}\pi n + \psi) \ ,$$

where $a = (1 + 2(0.22)^2)^{-1/2}$, $c = (0.22)^{-1}$, and $\psi = \arctan \{-[(0.22)\sqrt{2}]^{-1}\}$.

(c) Use the result of part (b) to conclude that the ellipse

$$x^2 + \frac{(v - c)^2}{2} = a^2$$

contains the limit set of the Poincaré map.

12. Using a numerical scheme such as Runge–Kutta or a software package, calculate the Poincaré map for equation (7) when $b = 0.3$, $\gamma = 1.2$, and $F = 0.2$. (Notice that the closer you start to the limiting point, the sooner the transient part will die out.) Compare your map with Figure 5.46(a) on page 302. Redo for $F = 0.28$.

13. Redo Problem 12 with $F = 0.31$. What kind of behavior does the solution exhibit?

14. Redo Problem 12 with $F = 0.65$. What kind of behavior does the solution exhibit?

15. Chaos Machine. Chaos can be illustrated using a long ruler, a short ruler, a pin, and a tie tack (pivot). Construct the double pendulum as shown in Figure 5.51(a). The pendulum is set in motion by releasing it from a position such as the one shown in Figure 5.51(b). Repeatedly set the pendulum in motion, each time trying to release it from the same position. Record the number of times the short ruler flips over and the direction in which it was moving. If the pendulum was released in *exactly* the same position each time, then the motion would be the same. However, from your experiments you will observe that even beginning close to the same position leads to very different motions. This double pendulum exhibits chaotic behavior.

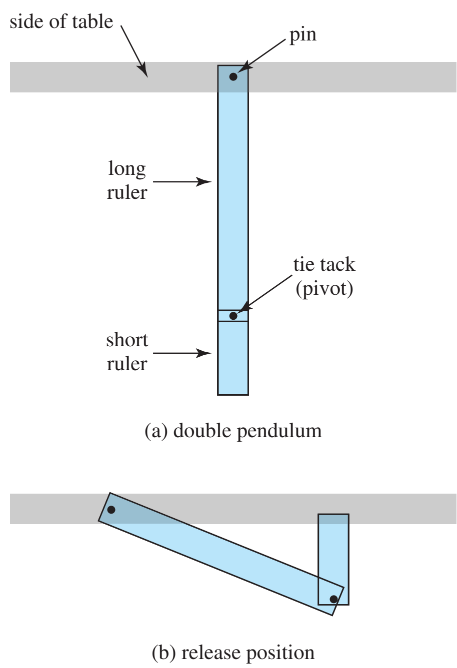


Figure 5.51 Double pendulum as a chaos machine